Optimal Portfolio Inputs: Various Methods

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Abstract:

In this document, I will model and back test our portfolio with various proposed models. It goes without saying that the portfolio with the greatest out-of-sample performance will be used for our current and future portfolio. The out-of-sample performance will be evaluated through a utility loss value derived in the introduction as well as through a portfolio performance comparison against the S&P 500 index and 1/N Portfolio rule. I found that the industry and constant correlation model tends to work best in out-of-sample performance.

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Introduction

The evaluation for each model is mainly motivated through the methods worked in Kan and Zhou (2007). The main portfolio problem is deciding on appropriate weights for securities in a portfolio to maximize return and minimize risk. Markowitz (1952)[1] has shown that for a given level of return generated through a Portfolio, there exists a set of weights that minimizes risk.

Let μ be a $N \times 1$ matrix of asset expected returns and Σ be a $N \times N$ covariance matrix of asset returns. Thus, the portfolio expected return is $\mu_p = w'\mu$ and portfolio risk (variance) is $\sigma_p^2 = w'\Sigma w$ where w is a $N \times 1$ vector of portfolio weights. In a perfect world, both parameters (μ & Σ) are known and the investors allocate their portfolio according to their risk preference or using the Power Utility function:

$$U(w) = \mu_p - \frac{\lambda}{2}\sigma_p^2$$

Where λ is an investors' risk aversion coefficient. Under optimality, the weights (w^*) are allocated as such:

$$w^* = \frac{1}{\lambda}\Sigma^{-1}\mu$$

Since both parameters aren't observable, we need to estimate them ($\hat{\Sigma}$ & $\hat{\mu}$). Denote our optimal portfolio weights with estimated inputs as $\widehat{w}^*$. Many methods have been proposed in improving future estimates of these two inputs and we will look at them in this report. To evaluate these methods, Kan and Zhou (2007) proposed a loss function:

$$L(w, \widehat{w}^*) = U(w^*) - U(\widehat{w}^*)$$

This function has a simple and intuitive meaning where the first term is the ex-post optimized portfolio's utility and the second term is the estimated ex-ante optimal portfolio out-of-sample performance. Since the first term is theoretically optimal, almost all out-of-sample performance is suboptimal and thus, $L(w, \widehat{w}^*)$ is strictly positive. Our main objective is to minimize this loss function toward zero.

The rest of this paper will be outlined as such: the methodology, introduction to various methods and estimated optimal weighting and finally, table and chart of performance measures.

Methodology

We will evaluate these models through sourcing historical estimates from 1/1/2005 – 1/1/2010 and testing out-of-sample performance from 1/1/2010 – 1/1/2014. Monthly returns will be used. The ETFS XLS and XPH will be excluded due to insufficient data for a total of N = 20 securities. Excel Solver with GRG non-linear optimization will be used since there is no closed-form solution for below constraints. Returns are calculated as the risk premium where $r_t = R_t - R_f$

[1] Portfolio Selection Harry Markowitz The Journal of Finance, Vol. 7, No. 1. (Mar., 1952), pp. 77-91.

and the Risk free rate is the U.S 10 Year Treasury Bond Ask Yield lagged by one period. Lastly, Dividends are excluded for the simplicity of data arrangements and calculations. All percentages represented have already been multiplied by 100.

To optimize, we set lambda equal to 1 and create two simple constraints (fully allocated portfolio and No short sale):

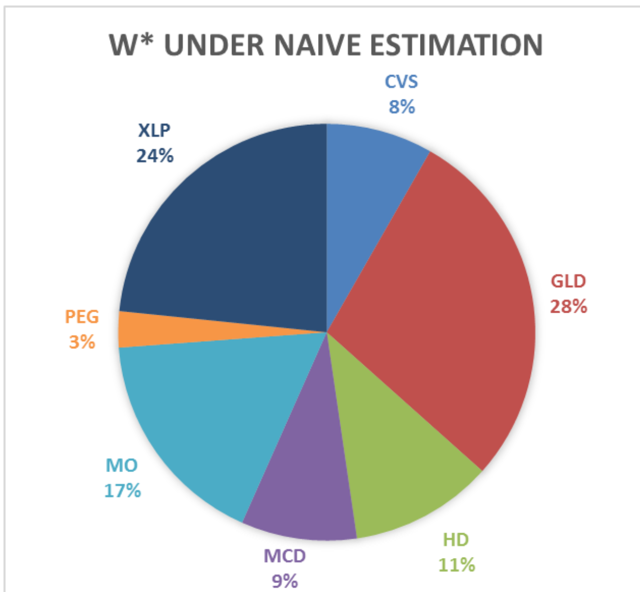
$$w^{*\prime}1 = 1$$

$$w_i^* \geq 0 \quad \forall\, i..N$$

Naïve Model

The Naïve Model is the benchmark model that we are going to test others against. The Naïve model makes the assumption that future parameters is the same as the historical estimate. Through countless research, this is not the case but we are going to include it anyway to gauge the magnitude of improvements from other models.

We obtain the following weights for the Portfolio:



All other stocks have a weighting of zero. This is primarily because (under the constraint of short-sale):

1. Strong positive covariance provide no benefits in diversification
2. From (1), stocks are then ranked by historical return to risk

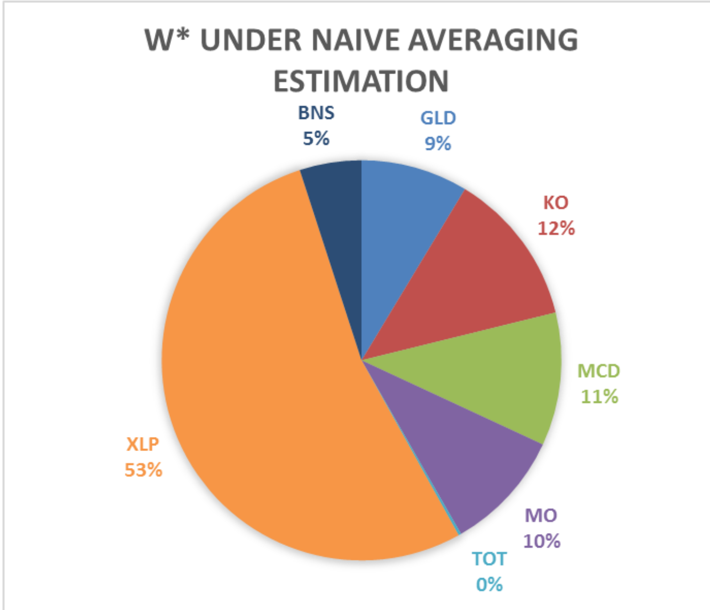
Averaging Models

For the next two set of results, only the correlations between equities are changed. The mean return premium and variance will remain as the naïve model forecast.

Naïve Averaging

In Naïve Averaging, all pairwise correlations are set equal to the average of all pairwise correlations in the matrix.

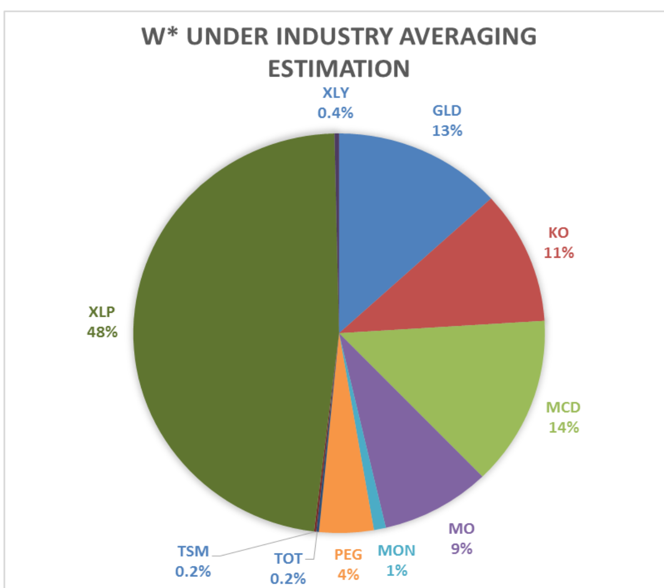
Optimizing across the estimate period we come upon the below weighting:



Industry Averaging

In industry averaging, we aggregate securities into groups by industry effects. The correlations within each group and with other groups are averaged and set. In The Fund's portfolio, we divide up our 20 stocks into 8 main industries as specified by GIC Classifications. They are Consumer Discretionary, Consumer Staples, Energy, Financials, Industrials, Information Technology, Materials and Utilities with 3, 4, 2, 3, 3, 2, 2, 1 stocks in each sector respectively.

Optimizing across these averages, we find a portfolio similar to the Naïve averaging model. However, there is a lot more non-zero weighted securities within the portfolio:



Single-Index Model

Pioneered by Sharpe, the single-index model (SIM) reduces the number of estimation inputs by assuming that all relevant securities have a return generating process dominated by one single index or:

$$r_{t,i} = \alpha_i + \beta_i r_{t,I} + \epsilon_t \quad \forall\, i$$

$$\epsilon_t \sim i.i.d(0, \sigma_e^2)$$

Where r_t is the risk premium generated at time T and by construction ϵ_t is an i.i.d noise process. Since population values of Alpha and Beta cannot be directly observed, we need to once again estimate them through regression analysis. Taking our estimates $\hat{\alpha}, \hat{\beta}$, we can calculate the covariance matrix and mean return vector:

$$\Sigma = \widehat{\mathrm{B}}\widehat{\mathrm{B}}' \sigma_I^2 + \sigma_e^2$$

$$\mu = \widehat{\mathrm{A}} + \widehat{\mathrm{B}}\mu_I$$

Where $\widehat{\mathrm{B}}$ is an $N \times 1$ vector of estimated Beta, $\widehat{\mathrm{A}}$ is an $N \times 1$ vector of estimated Alpha, σ_e^2 is a diagonal $N \times N$ matrix of unsystematic variance uncaptured by the SIM, μ_I & σ_I are the mean returns and variance of the index respectively. In financial time-series, both beta and alpha are subject to estimation uncertainty as the nature of the firm and the economic environment change. In the next two sub-sections, we will measure out-performance through both unadjusted beta and adjusted beta. Alpha, under the Capital Asset Pricing Model, is assumed to be zero and any indifference is due to sampling bias. Furthermore, it is extremely hard to predict alpha and we will make the prior assumption that it is zero.

Unadjusted Beta Estimation

In this section, we will take the plug-and-play approach to estimating beta. We will use the S&P 500 as the single-index since it contains many stocks within our portfolio. Furthermore, Mean Return of the S&P 500 input will be the historical geometric premium from Damodaran (2014) between 2004 and 2013.

Using the optimizer, we come upon a rather peculiar results where our portfolio is optimized when we put 100% of the portfolio weight into Gold Miners ETF. This is most likely because that since all pairwise covariance is generated through a single-index, the optimizer tends to rank each stock by their return and volatility against the market and put all its weight toward that single stock.

Adjusted Beta Estimation

Since Betas and Alphas vary over time for securities, this estimation risk can be reduced through adjusting the current Beta for the next forecast period. Elton, Gruber et al. (2014)[2] summarizes the studies done on Beta adjustments and found two significant study of Blume Adjustment and Vasicek Adjustment. Klemkosky and Martin (1975) concluded that

[2] Elton, Edwin J., Martin J. Gruber, Stephen J. Brown, and William N. Goetzmann. Modern portfolio theory and investment analysis. John Wiley & Sons, 2014.

the Vasicek Adjustment tend to outperform the Blume Adjustment methodology and dominate the Naïve method of estimation as well. In this study, we will specifically look at the Vasicek adjustment to our Beta portfolio.

The Vasicek adjustment is a Bayesian method in improving estimate through assuming a prior distribution for the Beta of each stock. In our application, we will assume a prior of historical industry beta and adjust each stock's beta to be a weighted average of the historical industry beta and the stock's sampled beta. Weighting is determined by the relative Standard Error of the Beta estimator in the linear regression. Higher the error, the more the Beta is adjusted toward the industry standard.

$$\hat{\beta}_i = \frac{\overline{Var(\hat{\beta})}}{Var(\hat{\beta}_i) + \overline{Var(\hat{\beta})}}(\hat{\beta}_i) + \frac{Var(\hat{\beta}_i)}{Var(\hat{\beta}_i) + \overline{Var(\hat{\beta})}}(\beta_{prior})$$

Where $Var(\hat{\beta}_p)$ is the Average Variance of Stock Betas that decreases as N increases and $Var(\hat{\beta}_i)$ is the Variance of the Beta of ith stock.

Optimizing the portfolio, the weighting is now 100% on stock PEG. This is in-align with previous explanation for 100% in Gold Miners ETF. However, the difference is that this portfolio significantly produces better out-of-sample performance

Fama & French Three-Factor Model

Fama and French (1992) proposes adding two additional cross-sectional variable to the CAPM to improve explanatory power of the model in individual stock returns. The variables they propose are Market Equity and Price to Book ratios and found that high value, small companies tend to outperform in expected returns. The variables are imposed in the regression through tracking portfolios of Small over Big Market equity and Low over High Price to Book ratio.

$$r_t = \alpha + \beta_1 r_m + \beta_2 r_{smb,t} + \beta_3 r_{hml,t} + \epsilon_t$$

In this section, I will model the portfolio on these characteristics to obtain better covariance and expected return estimates. Expected Returns of each stock will be estimated as the output from the three factor model using the sample estimation. The data for the tracking portfolio SMB and HML are retrieved from Kenneth French's data library.

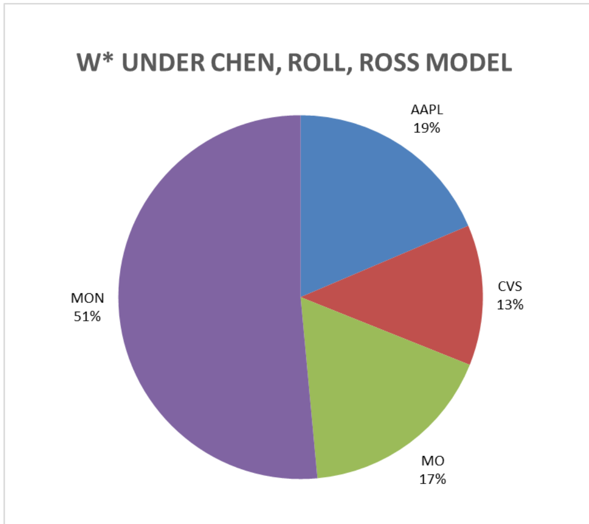
Feeding the numbers into the optimizer, we come upon an optimal weight of 78% in Gold Miners ETF and 22% in Home Depot.

Chen, Roll, Ross Model

Chen, Roll and Ross (1986) developed a fundamental index model where the variables in the return-generating process can be explained by economic factors. Their work is based on the financial theory that all stocks are priced from a consensus of expectations for future conditions. Any source of surprise as new information emerges will adjust the price of the stock. Similar to the structure of Arbitrage Pricing Theory, we will hypothesize a small set of significant economic factors that will adjust stock returns.

- Difference in yield from long-term U.S government bonds and long-term corporate bonds
- Difference in term structure through long-term U.S government bonds and short-term U.S treasury bills
- Difference in Actual and Expected Monthly Change in Inflation
- Difference in Actual and Expected Monthly Change in US Production
- Market Returns (S&P 500)

Regressing the factors with each stock return and optimizing the portfolio with sampled mean return and factor variance, we come upon the optimal portfolio.



Loss Performance

Below is a table of all models and their out-of-sample performance in comparison to the optimal utility. Note that the optimal Utility in the test sample is -2.25508.

Model	Mean Return	Variance	U(W*)	Utility Loss
Industry Averaging	0.711590277	6.869401	-2.72311	0.468032
Constant Averaging	0.773140317	7.120551	-2.78714	0.532057
Naïve Model	0.890076122	8.331002	-3.27542	1.020296
S&P 500 Index	0.92955556	16.5262627	-7.33358	5.078496
Single-Index (Adj)	-0.20882026	15.5459	-7.98177	5.726692
Fama & French	0.56386309	20.64006	-9.75617	7.501088
Chen, Roll Ross	1.295786306	30.94118	-14.1748	11.91972
Single-Index (Unadj)	0.109415755	31.10052	-15.4408	13.18577

It is clear that industry averaging provides the best out of sample result with only a utility loss of 0.468. This is consistent with other findings such as Elton and Gruber (2006). Similarly, multi-factor models can actually underperform a single factor adjusted model. This can be primarily attributed to great in-sample performance but poor out-of-sample due to estimation error as well as input forecast errors.

Portfolio Performance

Much research recently has found that often an equally weighted portfolio often outperform various optimization techniques. We will call this the 1/N rule and see how our optimized portfolio has performed compared to 1/N in terms of return, variance and Sharpe ratio. The Sharpe ratio will be defined as

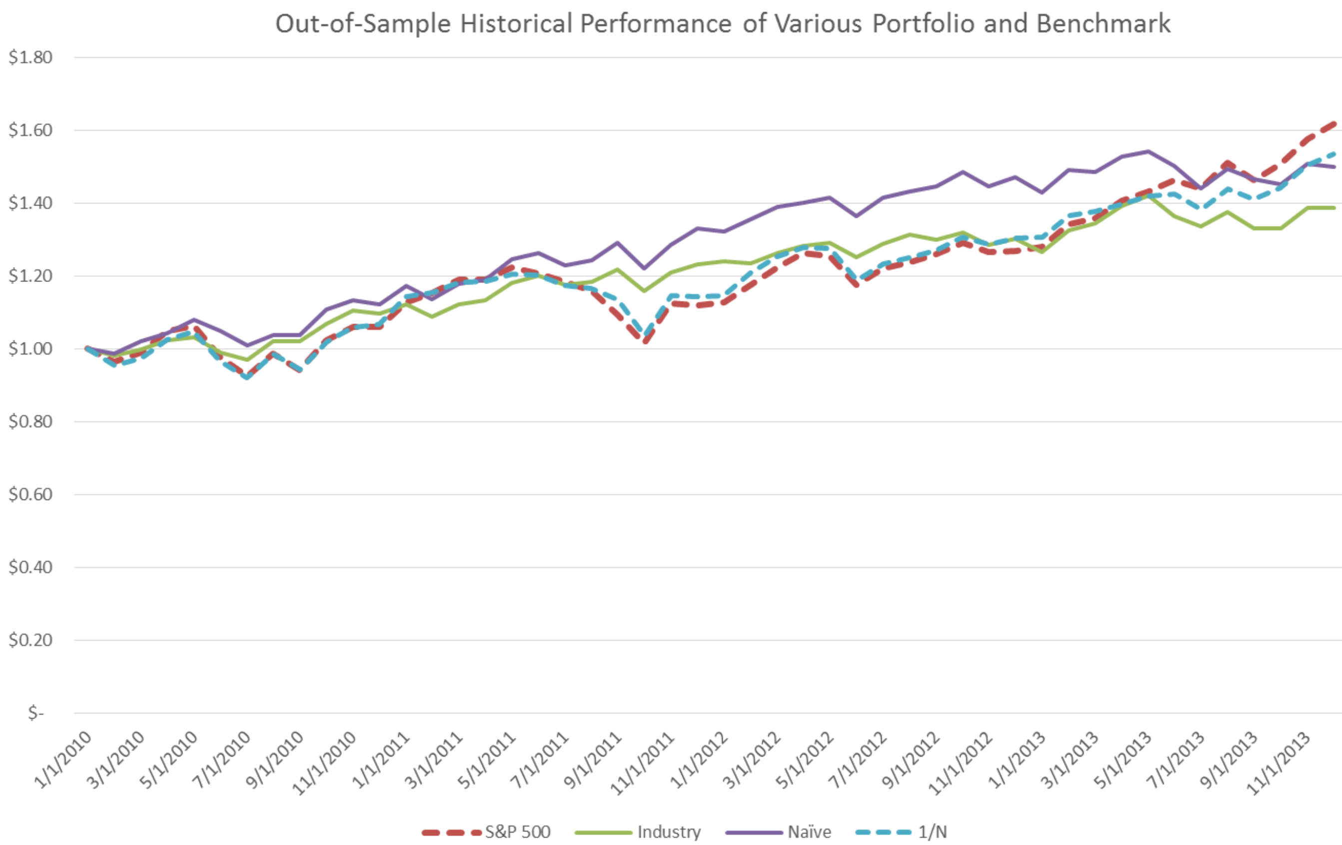
$$S_r = \frac{\overline{R_p} - \overline{R_f}}{\sigma_p}$$

Where $\overline{R_f}$ is the mean 10-year U.S risk-free yield between 2010-2014 (0.21%).

Model	Mean Return	Variance	Sharpe
Naïve Model	0.890076122	8.331002	0.308375
Constant Averaging	0.773140317	7.120551	0.289735
Industry Averaging	0.711590277	6.869401	0.2715
1/N	0.998062222	15.02024	0.257525
Chen, Roll Ross	1.295786306	30.94118	0.232951
S&P 500 Index	0.92955556	16.5262627	0.228659

Fama & French	0.56386309	20.64006	0.124113
Single-Index (Unadj)	0.109415755	31.10052	0.01962
Single-Index (Adj)	-0.20882026	15.5459	-0.05296

It is clear that through both performance measures, the industry, constant and naïve model all out-perform the single/multi-factor models. Below is a historical out-of-sample risk premia performance of the top optimized portfolio in comparison with 1/N and S&P 500.



Some important things to note:

1. The 1/N portfolio tends to track the S&P 500 closely, this is expected as usual since our portfolio is a large 20 stock portfolio. The 1/N portfolio outperformed the S&P 500 until midst 2013
2. The Naïve model out-performed all other measures between 2011 and 2012. This can primarily be attributed to its large 28% investment in gold during the bull rally and its heavy upscaling in specific consumer staples.
3. Industry model performed the worst but has the least volatility. It is mainly invested in consumer staples, scaled down on gold and took positions in the energy sector. It outperformed the benchmark and 1/N portfolio until Gold prices started declining.

Conclusion

Through this study, I have concluded that, in align with previous literature, the constant correlation and industry correlation model are best performers in portfolio optimization and modelling future covariance matrix. Furthermore, the factor models tend to perform poorly in out of sample estimation techniques mainly due to parameter estimate errors and future forecast errors. Far more research is needed to improve portfolio optimization such as extra-market covariance modelling from firm characteristics, analyst consensus from The Fund, hybrid models, shrinkage of different efficient frontier choices, Black-Litterman model, industry separate optimization, Sharpe Ratio optimization, etc. This paper offered a preliminary survey of available models to help with portfolio managers improve out-of-sample performance.